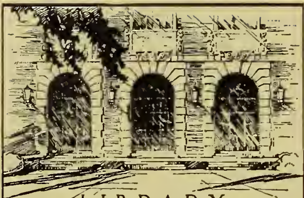

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ON THE SOLUTION OF A CLASS OF DUAL INTEGRAL EQUATIONS

by

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CONTENTS

ON THE SOLUTION OF A CLASS OF DUAL INTEGRAL EQUATIONS

1. Introduction

The purpose of this paper is to discuss the solution of a class of dual integral equations which appear in the formulation of electristatic and electromagnetic boundary value problems possessing circular symmetry. Two classes of equations are discussed out of which the first one admits a closed form solution. A Fredholm's equation of the second kind is derived for the second class and iterative means of solution are suggested. Some particular advantages of the formulation are pointed out.

2. The Dual Integral Equations - A.

The equations under discussion are:

$$\int_0^\infty \lambda^a f(\lambda)\, J_\mu(\lambda\rho)\, d\lambda = \int_0^\infty H(\lambda)\, J_\mu(\lambda\rho)\, d\lambda \qquad 0<\rho<1 \qquad (1)$$

$$\int_0^\infty f(\lambda)\, J_\mu(\lambda\rho)\, d\lambda = 0 \qquad \rho>1 \qquad (2)$$

where $H(\lambda)$ is a given function $a>-1-2\mu$ and $\mu\geq 0$.

Equations similar to these have been discussed by Tranter[1], Noble[2] and others. A method for obtaining a solution of the above equations in a close form will be given in the following section. The method outlined will be developed along a line which initially is somewhat similar to that of Tranter's. The chief advantages of the present method over that of Tranter's are the following:

(i) The solution is obtained in a closed form rather than in a series form given by Tranter.

(ii) The present method is especially suited for obtaining a Fredholm's equation of second kind for the dual equations - B, whereas Tranter's method involves the solution of an infinite set of equations for this case.

The equations B are:

$$\int_0^\infty \lambda^a \left\{1+T(\lambda)\right\} f(\lambda)\, J_\mu(\lambda\rho)\,d\lambda = \int_0^\infty H(\lambda)\, J_\mu(\lambda\rho)\,d\lambda \qquad 0<\rho<1 \qquad (3)$$

$$\int_0^\infty f(\lambda)\, J_\mu(\lambda\rho)\,d\lambda = 0 \qquad \rho>1 \qquad (4)$$

Where $T(\lambda)$ is a known function and $T(\lambda) \to 0$, or to a constant for large positive values of λ. The equations (3) and (4) are of considerable interest and will be the subject of discussion of a later section.

3. Method of Solution - Set A

As a first step assume that $f(\lambda)$ is represented by the series

$$f(\lambda) = \lambda^{1-k} \sum_{m=0}^{\infty} C_m\, J_{2m+\mu+k}(\lambda), \qquad k>0 \qquad (5)$$

Such a choice suggested by Tranter is in view of the relation

$$\int_0^\infty \lambda^{1-h}\, J_{\nu+2m+h}(\lambda)\quad J_\mu(\lambda\rho) = 0 \qquad \rho>1 \qquad (6)$$

The representation of $f(\lambda)$ in (5) automatically makes it satisfy (2). We still have to satisfy (1).

The first step toward this consists of multiplying both sides of (1) by

$$\rho^{\mu+1}\,(1-\rho^2)^{k-1}\,\mathcal{J}_p\,(k+\mu,\ \mu+1,\ \rho^2)$$

and integrating w.r.t. ρ in the range 0 to 1 where

$$\mathcal{J}_p^{(a,\gamma,x)} = {}_2F_1\,(-m,\ a+m;\ \gamma;m)$$

= Jacobi polynomial of order 'p' and F

= Hypergeometric function.

Upon carrying out the integration, one obtains by using the relation (see Tanter p. 113).

$$\int_0^1 \rho^{\mu+1} (1-\rho^2)^{k-1} \mathcal{J}_p J_\mu (\lambda\rho)\, d\rho = \lambda^{-k} J_{\mu+2m+k}(\lambda) \tag{7}$$

in (1), the following equation for $f(\lambda)$:

$$\int_0^\infty \lambda^{a-k} f(\lambda) \mathcal{J}_{2p+\mu+k} (\lambda)\, d\lambda = \int_0^\infty H(\lambda)\lambda^{-k} J_{2p+\mu+k}(\lambda)\, d\lambda \tag{8}$$

Substituting for $f(\lambda)$ from (5) there is obtained

$$\int_0^\infty [\lambda^{a+1-2k} \Sigma\, C_m J_{2m+\mu+k}(\lambda)]\, J_{2p+\mu+k}(\lambda)\, d\lambda = \int_0^\infty H(\lambda)\, \lambda^{-k} J_{2p+\mu+k}(\lambda)\, d\lambda \tag{9}$$

Now choose k such that

$$a + 1 - 2k = -1 \qquad \text{or} \qquad k = 1 + \frac{a}{2}. \tag{10}$$

Since it has been assumed that $a \geq -1 - 2\mu$, it follows that

$$k \geq \frac{1}{2} - \mu \qquad \text{or} \qquad \mu + k \geq \frac{1}{2}\,. \tag{11}$$

When k is chosen according to (10), (9) becomes

$$\int_0^\infty [\lambda^{-1} \sum_{m=0} C_m J_{2m+\mu+k}(\lambda)]\, J_{2p+\mu+k}(\lambda)\, d\lambda = \int_0^\infty H(\lambda)\, \lambda^{-(1+a/2)} J_{2p+\mu+k}(\lambda)\, d\lambda. \tag{12}$$

Now use the orthogonality relation for the Bessel functions, viz.,

$$\int_0^\infty \lambda^{-1} J_{2m+\mu+k}(\lambda)\, J_{2p+\mu+k} (\lambda)\, d\lambda = \frac{\delta_m^b}{(4m+2k)}$$

where

$$\delta^b_m = 0 \qquad \text{for} \qquad b \neq m$$

$$= 1 \qquad \text{for} \qquad b = m,$$

and obtain from (12)

$$C_p = \int_0^\infty H(\lambda)\, \lambda^{-(1+\alpha/2)}\, J_{2p+\mu+k}(\lambda)\, d\lambda \tag{13}$$

Letting $\nu = \mu+k+1$ and

$$S_\nu(\lambda) = \sum_{p=0}^{\infty} C_p\, J_{2p+\mu+k}(\lambda)\, d\lambda$$

we write $f(\lambda)$ given by (5) as

$$f(\lambda) = \lambda^{\mu-\nu}\, S_\nu(\lambda) \tag{14}$$

Following Wilkins[3], a closed form expression for $S_\nu(\lambda)$ is given by

$$S_\nu(\lambda) = H(\lambda)\lambda^{-\alpha/2} - \lambda \int_1^\infty r\, J_\nu(\lambda r) dr \int_0^\infty H(t) t^{-\alpha/2}\, J_\nu(tr)\, dt \tag{15}$$

provided the conditions specified by Wilkins on $H(\lambda)\, \lambda^{-\alpha/2}$ are met and $\nu \geq -\frac{1}{2}$. The condition on ν is already satisfied because of the condition expressed in (11). This is also the reason for imposing the condition on α which was stated in connection with the equations (1) and (2).

From (14) and (15) a close form representation of the solution is then

$$f(\lambda) = \lambda^{(\mu-\nu-\alpha/2)}\, H(\lambda) - \lambda^{\mu-\nu} \int_1^\infty r\, J_\nu(\lambda r) dr \int_0^\infty H(t) t^{-\alpha/2}\, J_\nu(tr)\, dt \tag{16}$$

Recalling that

$$\nu = \mu+k-1 = \mu+a/2$$

one can rewrite (16) as

$$f(\lambda) = \lambda^{-a} H(\lambda) - \lambda^{1-a/2} \int_1^\infty r\, J_{\mu+a/2}(\lambda r)dr \int_0^\infty H(t)t^{-a/2} J_{\mu+a/2}(tr) \quad (17a)$$

Since $\mu+a/2 = \nu \geq -1/2$ one may derive using Hankel's formula

$$\int_1^\infty r\, J_{\mu+a/2}(\lambda r)dr \int_0^\infty H(t)t^{-a/2} J_{\mu+a/2}(tr)dt$$

$$= H(\lambda)\, \lambda^{-1-a/2} - \int_0^1 r\, J_{\mu+a/2}(ar)dr \int_0^\infty H(t)t^{-a/2} J_{\mu+a/2}(tr)dt .$$

Using this in (17a) one obtains an alternate form of the solution as

$$f(\lambda) = \int_0^1 r\, J_{\mu+a/2}(ar)dr \int_0^\infty H(t)t^{-a/2} J_{\mu+a/2}(tr)\, dt \quad (17b)$$

and one may use one or the other form whichever is convenient.

4. The Dual Equations - B

The equations under consideration in the present section are the same as (3) and (4) and are repeated here for convenience. They are:

$$\int_0^\infty \lambda^a \left\{1+T(\lambda)\right\} f(\lambda)\, J_\mu(\lambda\rho)d\lambda = \int_0^\infty H(\lambda)\, J_\mu(\lambda\rho)\, d\lambda \quad , \qquad 0<\rho<1 \quad (18)$$

$$\int_0^\infty f(\lambda)\, J_\mu(\lambda\rho)\, d\lambda = 0 \qquad \rho>1. \quad (19)$$

Following through the same procedure which was used to derive (9), one now gets, assuming the following series form for $f(\lambda)$, viz.,

$$f(\lambda) = \lambda^{1-k} \sum_{m=0}^{\infty} C_m J_{2m+\mu+k}(\lambda),$$

the equation:

$$\int_0^\infty [\lambda^{\alpha+1-2k} \left\{1+T(\lambda)\right\} \sum_{m=0}^{\infty} C_m J_{2m+\mu+k}(\lambda)] J_{2p+\mu+k}(\lambda)d\lambda = \int_0^\infty H(\lambda)\lambda^{-k} J_{2p+\mu+k}(\lambda)d\lambda. \tag{20}$$

Again choose $k = 1+\alpha/2$ and obtain from (20)

$$\int_0^\infty [\lambda^{-1} \left\{1+T(\lambda)\right\} \sum_m C_m J_{2m+\mu+k}(\lambda)] J_{2p+\mu+k}(\lambda)d\lambda = \int_0^\infty H(\lambda)\lambda^{-k} J_{2p+\mu+k}(\lambda)d\lambda \tag{21}$$

Since (21) is true for all $p = 0, 1, 2, \ldots$ etc., one can write

$$\lambda^{-1} \left\{1 + T(\lambda)\right\} \quad S_\nu(\lambda) = H(\lambda)\lambda^{-k} + \phi(\lambda) \tag{22}$$

where

$$S_\nu(\lambda) = \sum_m C_m J_{2m+\mu+k}(\lambda) = \sum_m C_m J_{2m+1+\nu}(\lambda), \qquad \nu = \mu+k-1.$$

$\phi(\nu)$ is a function which satisfies

$$\int_0^\infty \phi(\lambda) J_{2p+1+\nu}(\lambda)d\lambda = 0 \quad \text{for all } p = 0, 1, 2, \ldots. \tag{23}$$

A suitable representation for $\phi(\lambda)$ which is suggested by the form of (15), is

$$\phi(\lambda) = \int_1^\infty J_\nu(\lambda x) \Psi(x)\, dx \tag{24}$$

which satisfies (23) as is easily seen by the substitution of (24) in (23) and the employment of the relation

$$\int_0^\infty J_{2p+1+\nu}(\lambda)\; J_\nu(\lambda x)\; dx = 0 \quad \text{for } x > 1. \tag{25}$$

It has been assumed in the above argument that it is possible to change the order of the integrations involved. Putting (24) in (22) we get after rearranging

$$S_\nu(\lambda) = \frac{H(\lambda)\lambda^{1-k}}{1+T(\lambda)} + \frac{\lambda}{1+T(\lambda)} \int_1^\infty J_\nu(\lambda x)\; \Psi(x) dx \tag{26}$$

Now, in view of (25),

$$\int_0^\infty S_\nu(\lambda)\; J_\nu(\lambda t) = \int_0^\infty \Sigma\, C_m\; J_{2m+1+2}(t)\;\; J_\nu(t) = 0 \qquad t > 1.$$

Hence, we can derive the following equation from (26):

$$-\int_0^\infty \frac{H(\lambda)\lambda^{1-k}}{1+T(\lambda)}\; J_\nu(\lambda t) d\lambda = \int_0^\infty \frac{\lambda}{1+T(\lambda)}\; J_\nu(\lambda t) d\lambda \int_1^\infty J_\nu(\lambda x)\; \Psi(x) dx\;, \quad t > 1. \tag{27}$$

Using Hankel's formula one can arrive at the equation

$$\frac{\Psi(t)}{t} = \int_0^\infty \lambda J_\nu(\lambda t) \int_1^\infty J_\nu(\lambda x)\; \Psi(x)\; dx \qquad t > 1\;, \tag{28}$$

and then rewrite (27) as

$$-\int_0^\infty \frac{H(\lambda)}{1+T(\lambda)}\; \lambda^{-\alpha/2}\; J_{\mu+\alpha/2}(\lambda t)\; d\lambda = \frac{\Psi(t)}{t} - \int_0^\infty \frac{\lambda T(\lambda)}{1+T(\lambda)}\; J_{\mu+\alpha/2}(\lambda t) d\lambda$$

$$\times \int_1^\infty J_{\mu+\alpha/2}(\lambda x)\; \Psi(x)\; dx, \qquad t > 1\;. \tag{29}$$

Equation (29) is a Fredholm's equation of second kind for the unknown Ψ. The equation of the second kind, as is well known, has the advantage that iterative methods can be employed to obtain successive approximations of the unknown and is therefore preferable to the integral equation of the first kind in most instances.

Some of the particular advantages of the Equation (29) will be discussed later in connection with the examples.

5. Examples

In this section we shall illustrate the techniques of dealing with the dual integral equations with some examples.

(a) The electrified disk:

The first set of equations to be considered are the following:

$$\int_0^\infty \lambda^{-1} f(\lambda)\, J_o\, (\lambda\rho)\, d\lambda = (2/\pi)\, V \int_0^\infty \lambda^{-1/2}\, J_{1/2}\, (\lambda)\, J_o\, (\lambda\rho)\, d\lambda = V \quad \text{for} \quad 0 < \rho < 1 \tag{30}$$

$$\int_0^\infty f(\lambda)\, J_o\, (\lambda\rho) = 0 \qquad \text{for } \rho > 1 \tag{31}$$

These equations are associated with the problem of finding the potential of an electrified disk at a potential V (for details see Tranter[4], p. 50).

The equations are of class - A and we identify by comparison with (1) and (2),

$$\alpha = -1,\ \mu = 0 \text{ and } H(\lambda) = (2/\pi)\, V\, J_{1/2}(\lambda)\, \lambda^{-1/2}$$

From (17a) the solution is given by

$$f(\lambda) = (2/\pi)\, V\, J_{1/2}\, (\lambda)\lambda^{1/2} - (2/\pi)\, V\, \lambda^{3/2} \int_1^\infty r\, J_{-1/2}\, (\lambda r)\, dr \int_0^\infty J_{1/2}(t)\, J_{-1/2}(t\text{(} \tag{32}$$

from Sonine's integral formulas (see Magnus[5]) we have

$$\int_0^\infty J_{1/2}\, (t)\, J_{-1/2}(rt)\, dt = 0 \qquad \text{for } r > 1.$$

The second term in the r.h.s. of (32) is therefore zero and we thus have the solution in the simple form

$$f(\lambda) = (2/\pi)\ V\ J_{1/2}\ (\lambda)\lambda^{1/2} \tag{33}$$

which is in agreement with the result given by Tranter. The use of (17b) will of course produce the same solution.

(b) Coaxial conducting disks:

The dual equations associated with the problem of two charged coaxial conducting disks with radius a and spaced apart at a distance of 2d have been given by Love[6] and they have been discussed by Tranter[1] and Noble[2]. The equations are:

$$\int_0^\infty \lambda^{-1}(1+e^{-2\lambda d}) f(\lambda) J_o(\lambda\rho)\, d\lambda = \left(\frac{2}{\pi}\right) V \int_0^\infty H(\lambda) J_o(\lambda\rho)\, d\lambda = V(\rho) \qquad \text{for } 0 < \rho < a \tag{34}$$

$$\int_0^\infty f(\lambda)\ J_o\ (\lambda\rho) = 0 \qquad \text{for } \rho > a \tag{35}$$

where $H(\lambda)$ and $V(\rho)$ are known.

Comparing with (18) and (19) we identify

$$\alpha = -1, \quad T(\lambda) = e^{-2\lambda d}, \quad \mu = 0 \quad \text{and} \quad H(\lambda) = \left(\frac{2}{\pi}\right) V \ \frac{J_{1/2}(\lambda a)}{\lambda^{1/2}}$$

Because (35) is to be valid for $\rho > a$ rather than $\rho > 1$, which is true for (19), slight modifications are necessary before the formulas in Section 4 can be used.

For one thing the series for $f(\lambda)$ has to be of the form

$$f(\lambda) = \lambda^{1-k} \quad C_m\ J_{2m + \mu + k}\ (a\lambda) = \lambda^{1-k}\ S_\nu\ (\lambda a) \tag{36}$$

in order for it to satisfy (35).

An equation corresponding to (26) may be derived and it reads

$$S_\nu(a\lambda) = \frac{H(\lambda)}{1+T(\lambda)} \lambda^{1-k} + \frac{\lambda}{1+T(\lambda)} \int_a^\infty J_\nu(\lambda x) \Psi(x) dx \tag{37}$$

An integral equation for $\Psi(x)$ is obtained by multiplying both sides of (37) by $J_\nu(\lambda t)$ and integrating w.r.t. t in the range 0 to ∞. This yields

$$-\int_0^\infty \frac{H(\lambda)}{1+T(\lambda)} \lambda^{1-k} J_\nu(\lambda t) = \int_0^\infty \frac{\lambda}{1+T(\lambda)} J_\nu(\lambda t) d\lambda \int_a^\infty J_\nu(\lambda x) \Psi(x) dx \qquad t > a, \tag{38}$$

which after rearranging may be put in the form

$$-\int_0^\infty \frac{H(\lambda)}{1+T(\lambda)} \lambda^{-\frac{\alpha}{2}} J_{\mu+\frac{\alpha}{2}}(\lambda t) d\lambda = \frac{\Psi(t)}{t} - \int_0^\infty \lambda \frac{T(\lambda)}{1+T(\lambda)} J_{\mu+\frac{\alpha}{2}}(\lambda t) d\lambda \int_a^\infty J_{-1/2}(\lambda x) \Psi \qquad t > a \tag{39}$$

Substituting for α, $T(\lambda)$ etc. in (39), for the problem under consideration, we obtain the equation

$$-(2/\pi) V \int_0^\infty J_{1/2}(\lambda a) J_{-1/2}(\lambda t) (1+e^{-2\lambda d})^{-1} d\lambda$$

$$= \frac{\Psi(t)}{t} - \int_0^\infty \frac{\lambda e^{-2\lambda d}}{1+e^{-2\lambda d}} J_{-1/2}(\lambda t) d\lambda \int_0^\infty J_{-1/2}(\lambda x) \Psi(x) dx \qquad t > a \tag{40}$$

Equation (40) is the desired Fredholm's integral equation for $\Psi(x)$. Once $\Psi(x)$ is known for $x > a$, $f(\lambda)$ is found through the use of (36) and (37).

We shall now discuss several special cases, e.g.,

(i) $d \to 0$ (ii) $d \to \infty$ (iii) $a \to 0$, (iv) $a \to \infty$, in connection with (39) and (40).

(i) $d \to 0$:

If $d \to 0$, (40) reads in the limit,

$$-\left(\frac{2}{\pi}\right) V \int_0^\infty J_{1/2}(\lambda a) J_{-1/2}(\lambda t) d\lambda = \Psi \frac{(t)}{t} - \frac{1}{2} \int_0^\infty \lambda J_{-1/2}(\lambda t) d\lambda \int_a^\infty J_{-1/2}(\lambda x) \Psi(x) dx, \quad t \tag{41}$$

Since the integral in the l.h.s. of (41) is zero for $t > a$, the solution

$$\Psi(t) = o \qquad \text{for } t > a$$

is seen to satisfy (41). From (36) and (37) we have

$$f(\lambda) = \lambda^{1-k} S_\nu(a\lambda) = \frac{1}{2} H(\lambda) \lambda^{2-2k} = \frac{1}{\pi} V J_{1/2} (a\lambda) \lambda^{1/2} \tag{42}$$

which may be verified to be in agreement with the solution of (30) by comparing the equations (30) and (34) and their solutions.

(ii) $d \to \infty$

This is a trivial case since (34) becomes identical with (30) for $d \to \infty$.

(iii) $a \to 0$

When $a \to 0$, a look at (34) and (35) will tell us that $f(\lambda)$ should also tend to zero for any general $H(\lambda)$.

From (39), letting $a = 0$ we get after rearranging the r.h.s.

$$\int_0^\infty H(\lambda) \lambda^{1/2} J_\nu (\lambda t) (1+T(\lambda))^{-1} d\lambda$$

$$= \int_0^\infty \frac{\lambda}{1+T(\lambda)} J_\nu (\lambda t) dx \int_0^\infty J_\nu(\lambda x) \Psi (x) dx, \qquad t > o. \tag{4.3}$$

Using Hankel's inversion formula we get

$$-H(\lambda) \lambda^{1/2} (1+T(\lambda))^{-1} = \lambda(1+T(\lambda))^{-1} \int_0^\infty J_\nu(\lambda x) \Psi (x) dx \tag{4 4}$$

From (37) one has for $a = 0$

$$S_\nu(x) = 0 \qquad \text{implying } f(\lambda) = 0.$$

For small a, the equation (44) may be used for a starting approximation of $\Psi(\lambda)$ for the purposes of iteration.

(iv) $a \to \infty$:

Again considering an arbitrary $H(\lambda)$ and a very large radius of the disk, we see from (37) that

$$S_\nu (a\lambda) \to \frac{H(\lambda)}{1+T(\lambda)} \lambda^{1-h} \tag{4.5}$$

Now since the range of validity of (34) is now the entire positive range of ρ, it is easily seen from that equation, that

$$\lambda^{-1} f(\lambda) = S_\nu \lambda^{-h} \to \frac{H(\lambda)}{1+T(\lambda)} \quad \text{or } S_\nu \to \frac{\lambda^k H(\lambda)}{1+T(\lambda)} \tag{46}$$

for $a = x - 1$. Since $k = 1 + \frac{\alpha}{2}$, (45) and (46) are seen to be in agreement. Again (46) may be used for a starting approximation for iteration when α is large.

The Fredholm's equation (39) is therefore useful from the point of view of iterating both for large and small radii of the disks. This is found to be of great advantage when applying the technique outlined here to the problem of electromagnetic diffraction due to a circular disk or a hole for both large and small diameters. In addition to this, through the use of techniques developed here, a Fredholm's integral equation may also be derived which is particularly suitable for small radius α because the kernel of this parallel equation is small when α is small. The detailed discussion on some of these topics will appear in a later report.

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by

Robert L. Carrel

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